

The quantum brachistochrone problem for an arbitrary spin in a magnetic field

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We consider quantum brachistochrone evolution for a spin- s system on rotational manifolds. Such manifolds are determined by the rotation of the eigenstates of the operator of projection of spin- s on some direction. The Fubini-Study metrics of these manifolds are those of spheres with radii dependent on the value of the spin and on the value of the spin projection. The conditions for optimal evolution of the spin- s system on rotational manifolds are obtained.

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1 Introduction

The interest in quantum brachistochrone problem has increased after the publication of paper [1], where Carlini et al. considered the following problem: What is the optimal Hamiltonian, under a given set of constraints, such that the evolution from a given initial state $|\psi_i\rangle$ to a given final one $|\psi_f\rangle$ is achieved in the shortest time? Using a variational principle, the authors of this work solved the brachistochrone problem for some specific examples of constraints. In [2], results analogous to those of [1] were obtained more directly due to symmetry properties of the quantum state space. The authors of this paper showed that the brachistochrone evolution between two states $|\psi_i\rangle$ and $|\psi_f\rangle$ which are set in the Hilbert space of dimension n is reduced to the evolution on the two-dimensional subspace spanned by the two vectors $|\psi_i\rangle$ and $|\psi_f\rangle$.

It is easy to find a geodesic between two quantum states and optimal Hamiltonian for a two-level system with a given set of constraints. The quantum brachistochrone problem for a such system was considered in many papers (see, for example, [3, 4, 5, 6]). There are multilevel physical systems with dimensionality higher than two whose properties do not allow reducing of quantum evolution to the evolution on the two-dimensional subspace as in the paper [2]. For example, the Hamiltonian of a spin- s system (where $s > 1/2$) in an external magnetic field contains only two free parameters, which define direction of the magnetic field. Dimensionality of the Hilbert space of this system is $2s + 1$. So an arbitrary state of this system must be defined by $4s$ real parameters. It means that we cannot provide evolution between two arbitrary states of a spin- s system with help of magnetic field.

The quantum brachistochrone evolution for a spin-1 system in a magnetic field was considered in [7].

Multilevel quantum systems with dimensionality higher than two could be more efficient than qubit, because they provide a way for more dense data recording. A three- and four-level systems are the simplest multilevel systems after a two-level system. In quantum information these systems are called *qutrit* and *ququad*, respectively. In general, a d -level quantum system is called *qudit*. The channel capacity for these systems is greater than for a two-level system [8]. The quantum cryptography protocols created by qudits are more secure against eavesdropping attacks than the cryptography protocols created by qubits [9, 10, 11, 12]. Therefore, qudits are more efficient in many problems of quantum computation [13, 14, 15] and quantum cryptography [11, 16]. Design of a qutrit quantum computer based on a trapped ion in the presence of magnetic field gradient is presented in [17]. This work is the generalization of [18, 19, 20], where design of a qubit quantum computer on trapped ion was considered. Another quantum system which is suitable for quantum computations with qutrits is a polarized biphoton [21, 22, 23, 24, 25], which is formed by two correlated photons.

A geometric approach to study qudit system has been developed in [26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. In [26, 27, 28, 29] it is shown that in the case of qubit systems, finding an optimal quantum circuit of a unitary operation is closely related to the problem of finding the minimal distance between two point on the Riemannian metric. A similar problem for the case of n qutrits was considered in [30]. The authors of this work showed that the optimal quantum circuit is equivalent to the shortest path between two points in a certain curved geometry of $SU(3^n)$.

In [32] the authors presented three different matrix bases that can be used to decompose density matrices of a d -dimensional quantum system. Namely, the generalized Gell-Mann matrix basis, the polarization operator basis, and the Weyl operator basis. These decompositions were identified with the Bloch vector for qudit which is the generalization of the well known qubit case. In [33] it was shown that physical characteristics of spin-1/2, spin-1, spin-3/2, and spin-2 systems can be represented by geometrical features that are preferentially identified on the complex manifold.

The geometrical properties of some wellknown coherent states manifolds, which are generated by an action of a Lie group on a fixed states, was studied in details in [34, 35]. In these articles the Fubini-Study metric of these manifolds was examined. The authors considered the atomic coherent states, generated by the action of the $SU(2)$ displacement operator on the eigenstate of the z -component of the angular momentum operator which corresponds to the lowest eigenvalue. It was shown that the metric of the manifold of this

state is that of the sphere.

In this paper, we consider quantum brachistochrone evolution of spin- s system on the manifolds determined by a rotation of the eigenstates of the operator of projection of spin- s on some direction. In Section 2 it is shown that two such manifolds exist for a spin-1 system. Each of them is defined by two real parameters. Also, we show that they do not intersect each other. The Fubini-Study metrics of these manifolds are obtained in Section 3. It is shown that these are the metrics of the spheres with radii dependent on the value of the spin and on the value of the spin projection. The quantum brachistochrone problem on each of the manifolds is considered in Section 4. We generalize this problem for an arbitrary spin s (Section 5). In Section 6 we give conclusions.

2 The rotational manifolds of spin-1/2 and spin-1 systems

The rotation of the quantum state of spin- s $|\psi_i\rangle$ through an angle χ about an axis in the direction of the unit vector \mathbf{n} can be realized as follows:

$$|\psi_f\rangle = e^{-i\chi\mathbf{S}\cdot\mathbf{n}}|\psi_i\rangle, \quad (1)$$

where \mathbf{S} is the spin- s operator. In the spherical coordinates the vector \mathbf{n} can be represented as follows $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where θ and ϕ are the polar and azimuthal angles, respectively. We set $\hbar = 1$. For instance, let us consider the rotation of the spin-1/2 system. In this case the spin-1/2 operator can be represented by the Pauli matrices $\boldsymbol{\sigma}$ as $\frac{1}{2}\boldsymbol{\sigma}$. Rotations through an angle θ about the y -axis and an angle ϕ about the z -axis allow us to achieve an arbitrary quantum state of spin-1/2 having started from the eigenvectors of σ_z

$$|\psi^+\rangle = e^{-i\frac{\phi}{2}\sigma_z}e^{-i\frac{\theta}{2}\sigma_y}|\uparrow\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}, \quad (2)$$

$$|\psi^-\rangle = e^{-i\frac{\phi}{2}\sigma_z}e^{-i\frac{\theta}{2}\sigma_y}|\downarrow\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}. \quad (3)$$

Here we use the fact that

$$e^{-i\frac{\theta}{2}\sigma_\alpha} = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\sigma_\alpha,$$

where $\alpha = x, y, z$. The states (2) and (3) are the eigenstates of the operator $\boldsymbol{\sigma}\cdot\mathbf{n}$ with 1 and -1 eigenvalues, respectively. Choosing parameters $\theta \in [0, \pi]$

and $\phi \in [0, 2\pi]$ in the equations either (2) or (3) we can achieve an arbitrary state of the spin-1/2 system. In other words, these states cover the entire state space of the spin-1/2 system.

Let us consider a similar problem for a spin-1 system. In the matrix representation components of the spin-1 operator read:

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4)$$

It is convenient to represent the operator which provides the rotation of the quantum state of spin-1 around vector \mathbf{n} in the form [7]

$$e^{-i\chi \mathbf{S} \cdot \mathbf{n}} = 1 - (\mathbf{S} \cdot \mathbf{n})^2 2 \sin^2 \frac{\chi}{2} - i \mathbf{S} \cdot \mathbf{n} \sin \chi. \quad (5)$$

The operator $\mathbf{S} \cdot \mathbf{n}$ has three eigenvalues 1, 0, -1 with the corresponding eigenvectors $|\psi_1\rangle$, $|\psi_0\rangle$, $|\psi_{-1}\rangle$. An arbitrary state of a three-level system can be written as a linear combination of these eigenvectors. It is enough to prove the equation (5) only for these eigenvectors. It is easy to verify that for a parameter λ , which takes only three values 1, 0 and -1 , we have

$$e^{\lambda x} = (1 - \lambda)(1 + \lambda) + \frac{1}{2}\lambda(\lambda + 1)e^x + \frac{1}{2}\lambda(\lambda - 1)e^{-x}. \quad (6)$$

Then, using (6) for the unitary operator of rotation, we obtain the relation (5). In general for the parameter λ , which takes n values, namely, $\lambda_1, \lambda_2, \dots, \lambda_n$, we have

$$e^{\lambda x} = \sum_{m \neq k=1}^n \prod_{k=1}^n \frac{\lambda - \lambda_k}{\lambda_m - \lambda_k} e^{\lambda_m x}. \quad (7)$$

Now, using the equation (5), we can represent the operators which provide the rotations of the quantum state of the spin-1 system around x -, y - and z -axis as follows

$$e^{-i\theta S_\alpha} = 1 - S_\alpha^2 2 \sin^2 \frac{\theta}{2} - i S_\alpha \sin \theta, \quad (8)$$

where $\alpha = x, y, z$. The eigenstates of S_z with the eigenvalues 1, 0, -1 we denote as follows: $|1\rangle$, $|0\rangle$, $|-1\rangle$. These eigenvectors play the role of the basis vectors. Let us consider the rotations of these eigenstates through angles θ and ϕ about the y - and z -axis, respectively. Then, using the equation (8),

we obtain the following states

$$|\psi_1\rangle = e^{-i\phi S_z} e^{-i\theta S_y} |1\rangle = \begin{pmatrix} \frac{1}{2}(1 + \cos\theta) e^{-i\phi} \\ \frac{1}{\sqrt{2}} \sin\theta \\ \frac{1}{2}(1 - \cos\theta) e^{i\phi} \end{pmatrix}, \quad (9)$$

$$|\psi_0\rangle = e^{-i\phi S_z} e^{-i\theta S_y} |0\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin\theta e^{-i\phi} \\ \cos\theta \\ \frac{1}{\sqrt{2}} \sin\theta e^{i\phi} \end{pmatrix}, \quad (10)$$

$$|\psi_{-1}\rangle = e^{-i\phi S_z} e^{-i\theta S_y} |-1\rangle = \begin{pmatrix} \frac{1}{2}(1 - \cos\theta) e^{-i\phi} \\ -\frac{1}{\sqrt{2}} \sin\theta \\ \frac{1}{2}(1 + \cos\theta) e^{i\phi} \end{pmatrix}. \quad (11)$$

It is important to note that these states are eigenstates of the operator $\mathbf{S} \cdot \mathbf{n}$ with the corresponding eigenvalues 1, 0 and -1 , respectively. From the analysis of these eigenstates it is clear that the states $|\psi_1\rangle$ and $|\psi_{-1}\rangle$ belong to the same rotational manifold and the state $|\psi_0\rangle$ belongs to another rotational manifold. To cover the entire manifold defined by the states $|\psi_1\rangle$ and $|\psi_{-1}\rangle$ it is enough that the parameters θ and ϕ belong to the intervals $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. In the case of the manifold defined by the state $|\psi_0\rangle$ we have that it is twice covered by the intervals $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ because the following substitutions $\theta \rightarrow \pi - \theta$ and $\phi \rightarrow \phi + \pi$ transform the state $|\psi_0\rangle$ into itself modulo a global phase. At the same time these substitutions allow us to transform the state $|\psi_1\rangle$ into the state $|\psi_{-1}\rangle$. However, it does not exist any substitution that transforms either the state $|\psi_1\rangle$ or $|\psi_{-1}\rangle$ into the state $|\psi_0\rangle$. It means that these manifolds do not intersect each other.

In contrast to the case of the spin-1/2 system, where the rotation manifold coincides with the two-dimensional quantum space, none of the manifolds defined by the states (9)-(11) coincides with the quantum space of the spin-1 system. The number of parameters which determine each of these manifolds is not sufficient to specify the quantum space of the spin-1 system which must be defined by four real parameters. Moreover, a linear combination of the states which belong to one of these manifolds does not belong to it.

3 The Fubini-Study metrics of the rotational manifolds of spin-1/2 and spin-1 systems

The Fubini-Study metric is the infinitesimal distance ds between two neighbouring pure quantum states $|\psi(\xi^\alpha)\rangle$ and $|\psi(\xi^\alpha + d\xi^\alpha)\rangle$ [6, 36, 37]. It is given by the following expression

$$ds^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta, \quad (12)$$

where ξ^α is a set of real parameters which define the state $|\psi(\xi^\alpha)\rangle$. The components of the metric tensor $g_{\alpha\beta}$ have the form:

$$g_{\alpha\beta} = \gamma^2 \Re(\langle \psi_\alpha | \psi_\beta \rangle - \langle \psi_\alpha | \psi \rangle \langle \psi | \psi_\beta \rangle), \quad (13)$$

where γ is an arbitrary factor which is often chosen 1, $\sqrt{2}$ or 2 and

$$|\psi_\alpha\rangle = \frac{\partial}{\partial \xi^\alpha} |\psi\rangle. \quad (14)$$

For instance, the Fubini-Study metric of the space of a spin-1/2 system, which is spanned by the states (2) and (3), reads [6, 37]

$$ds^2 = \frac{\gamma^2}{4} ((d\theta)^2 + \sin^2 \theta (d\phi)^2). \quad (15)$$

Here, the angles θ and ϕ play the role of the parameters ξ^α . Note that (15) is the metric of the sphere of radius $\gamma/2$. In case of $\gamma = 2$ we obtain the metric of the Bloch sphere (the Bloch sphere is a sphere of the unit radius which represents the state space of a two-level system). The states $|\psi^+\rangle$ (2) and $|\psi^-\rangle$ (3) correspond to the antipodal points on this sphere.

Now let us calculate metrics of the rotational manifolds defined by the states (9)-(11) obtained for $s = 1$. These states are also determined by two real parameters θ and ϕ . As we mentioned earlier, the eigenstates (9), (11) belong to the same manifold and eigenstate (10) belongs to another manifold. Therefore, in order to obtain the Fubini-Study metric of these manifolds it is enough to consider the eigenstates (9) and (10). Let us calculate the following derivatives from these eigenstates:

$$\begin{aligned} |\psi_{1\ \theta}\rangle &= \begin{pmatrix} -\frac{1}{2} \sin \theta e^{-i\phi} \\ \frac{1}{\sqrt{2}} \cos \theta \\ \frac{1}{2} \sin \theta e^{i\phi} \end{pmatrix}, & |\psi_{1\ \phi}\rangle &= \begin{pmatrix} -\frac{i}{2} (1 + \cos \theta) e^{-i\phi} \\ 0 \\ \frac{i}{2} (1 - \cos \theta) e^{i\phi} \end{pmatrix}, \\ |\psi_{0\ \theta}\rangle &= \begin{pmatrix} -\frac{1}{\sqrt{2}} \cos \theta e^{-i\phi} \\ -\sin \theta \\ \frac{1}{\sqrt{2}} \cos \theta e^{i\phi} \end{pmatrix}, & |\psi_{0\ \phi}\rangle &= \begin{pmatrix} \frac{i}{\sqrt{2}} \sin \theta e^{-i\phi} \\ 0 \\ \frac{i}{\sqrt{2}} \sin \theta e^{i\phi} \end{pmatrix}. \end{aligned} \quad (16)$$

Using these derivatives we obtain the following scalar products:

$$\begin{aligned}
\langle \psi_1 | \psi_{1\ \theta} \rangle &= 0, & \langle \psi_{1\ \theta} | \psi_{1\ \theta} \rangle &= \frac{1}{2}, \\
\langle \psi_1 | \psi_{1\ \phi} \rangle &= -i \cos \theta, & \langle \psi_{1\ \phi} | \psi_{1\ \phi} \rangle &= \frac{1}{2} (1 + \cos^2 \theta), \\
\langle \psi_{1\ \theta} | \psi_{1\ \phi} \rangle &= \frac{i}{2} \sin \theta, \\
\langle \psi_0 | \psi_{0\ \theta} \rangle &= 0, & \langle \psi_{0\ \theta} | \psi_{0\ \theta} \rangle &= 1, \\
\langle \psi_0 | \psi_{0\ \phi} \rangle &= 0, & \langle \psi_{0\ \phi} | \psi_{0\ \phi} \rangle &= \sin^2 \theta, \\
\langle \psi_{0\ \theta} | \psi_{0\ \phi} \rangle &= 0.
\end{aligned} \tag{17}$$

Substituting these products into the definition of the components of metric tensor (13), the Fubini-Study metrics of the rotational manifolds defined by the eigenstates (9)-(11) take the form:

$$ds_1^2 = \frac{\gamma^2}{2} ((d\theta)^2 + \sin^2 \theta (d\phi)^2), \tag{18}$$

$$ds_0^2 = \gamma^2 ((d\theta)^2 + \sin^2 \theta (d\phi)^2), \tag{19}$$

where subscript in ds indicates the eigenvalue that in turn indicates the manifold. It is easy to see that the expression (18) describes metric of the sphere of radius $\gamma/\sqrt{2}$. The orthogonal states correspond to antipodal points on this sphere. In the case of manifold which is defined by (19) we obtain another result. As we mentioned earlier, the substitutions $\theta \rightarrow \pi - \theta$ and $\phi \rightarrow \phi + \pi$ transform the state $|\psi_0\rangle$ (10) into itself modulo a global phase. The manifold defined by this state is called elliptic geometry. It is important to note that orthogonal states on the manifold (19) are separated by an angle $\pi/2$. Really, the scalar product of two states $|\psi_0\rangle \equiv |\psi_0(\theta, \phi)\rangle$ and $|\psi'_0\rangle \equiv |\psi_0(\theta', \phi')\rangle$, which belong to manifold (19), reads

$$\langle \psi_0 | \psi'_0 \rangle = \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'. \tag{20}$$

On the other hand, this is the scalar product of two unit vectors \mathbf{n} defined by the spherical angles θ, ϕ and \mathbf{n}' defined by the spherical angles θ', ϕ' , respectively. This product becomes zero when the angle between these vectors is $\pi/2$. This confirms our conclusion.

4 The quantum brachistochrone problem for spin-1 system in a magnetic field

In this section we consider quantum brachistochrone evolution on the rotational manifolds defined by the metrics (18), (19), obtained for spin-1 system.

Hamiltonian providing such evolution is the Hamiltonian of the spin-1 system in an external magnetic field directed along the unit vector \mathbf{n}'

$$H = \omega \mathbf{S} \cdot \mathbf{n}', \quad (21)$$

where ω is proportional to the strength of the magnetic field and is measured in frequency units, \mathbf{n}' is defined by two spherical angles θ' and ϕ' . As we mentioned above we set $\hbar = 1$. The eigenvalues of this Hamiltonian are ω , 0 and $-\omega$ with the corresponding eigenstates (9)-(11), where angles θ and ϕ are denoted as θ' and ϕ' , respectively. Hamiltonian (21) contains only two free parameters, namely, two angles θ' and ϕ' . The general state for a spin-1 system is defined by four real parameters. Therefore, we cannot reach an arbitrary state using the operator of evolution with Hamiltonian (21).

The quantum brachistochrone problem for spin-1 system in a magnetic field is studied in the paper [7]. The authors considered the following question: what is the optimal direction of the magnetic field \mathbf{n}' at the fixed value ω , such that the evolution from a given initial state $|\psi_i\rangle$ to a given final one $|\psi_f\rangle$ is achieved in the shortest time? In that paper, studying directly the evolution of quantum state with the Hamiltonian (21), conditions for optimal evolution were obtained. We solve this problem using geometric properties of manifolds defined by (18) and (19). Let us consider it in detail.

Using equation (6), the unitary operator of evolution with Hamiltonian (21) takes the form

$$e^{-iHt} = 1 - (\mathbf{S} \cdot \mathbf{n}')^2 2 \sin^2 \frac{\omega t}{2} - i \mathbf{S} \cdot \mathbf{n}' \sin \omega t. \quad (22)$$

Now, using this operator we can consider the quantum evolution of the system described by Hamiltonian (21). Let us take the initial states as the eigenstates of S_z $|1\rangle$, $|0\rangle$ and $|-1\rangle$ [7]. Then using (22), we finally find

$$|\psi_1(t)\rangle = e^{-iHt}|1\rangle = \begin{pmatrix} 1 - (1 + \cos^2 \theta') \sin^2 \frac{\omega t}{2} - i \cos \theta' \sin \omega t \\ - \left(\sqrt{2} \cos \theta' \sin \theta' \sin^2 \frac{\omega t}{2} + \frac{i}{\sqrt{2}} \sin \theta' \sin \omega t \right) e^{i\phi'} \\ - \sin^2 \theta' \sin^2 \frac{\omega t}{2} e^{i2\phi'} \end{pmatrix}, \quad (23)$$

$$|\psi_0(t)\rangle = e^{-iHt}|0\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} \left(2 \cos \theta' \sin \theta' \sin^2 \frac{\omega t}{2} + i \sin \theta' \sin \omega t \right) e^{-i\phi'} \\ 1 - 2 \sin^2 \theta' \sin^2 \frac{\omega t}{2} \\ \frac{1}{\sqrt{2}} \left(2 \cos \theta' \sin \theta' \sin^2 \frac{\omega t}{2} - i \sin \theta' \sin \omega t \right) e^{i\phi'} \end{pmatrix}, \quad (24)$$

$$|\psi_{-1}(t)\rangle = e^{-iHt}|-1\rangle = \begin{pmatrix} -\sin^2 \theta' \sin^2 \frac{\omega t}{2} e^{-i2\phi'} \\ \left(\sqrt{2} \cos \theta' \sin \theta' \sin^2 \frac{\omega t}{2} - \frac{i}{\sqrt{2}} \sin \theta' \sin \omega t \right) e^{-i\phi'} \\ 1 - (1 + \cos^2 \theta') \sin^2 \frac{\omega t}{2} + i \cos \theta' \sin \omega t \end{pmatrix}. \quad (25)$$

It is easy to show that the states (23)-(25) are equal to the eigenstates (9)-(11) modulo a global phase:

$$|\psi_1(t)\rangle = e^{i\beta}|\psi_1\rangle, \quad (26)$$

$$|\psi_0(t)\rangle = |\psi_0\rangle, \quad (27)$$

$$|\psi_{-1}(t)\rangle = e^{-i\beta}|\psi_{-1}\rangle, \quad (28)$$

where

$$\beta = 2\phi' - \phi + (2k + 1)\pi,$$

k is an arbitrary integer. Here we introduce the following notation:

$$\begin{aligned} \phi &= \phi' - \arctan \frac{\cos \frac{\omega t}{2}}{\cos \theta' \sin \frac{\omega t}{2}}, \\ \sin \frac{\theta}{2} &= \sin \theta' \sin \frac{\omega t}{2}. \end{aligned} \quad (29)$$

As we can see, if the initial state belongs to one of these manifolds then the quantum evolution of the system takes place on the same manifold. In other words, as we mentioned previously, Hamiltonian (21) realizes quantum evolution on two manifolds separately and does not mix them. For instance, we cannot achieve the state $|0\rangle$ starting from the state $|1\rangle$.

The period of time of evolution from the initial state $|\psi_i\rangle$ to the final one $|\psi_f\rangle$ is given by the ratio

$$t = \frac{s}{v}, \quad (30)$$

where s is the path length of evolution between these states and v is the speed of evolution. The shortest path joining the two states on the sphere is the length of the great circle arc (the length of the geodesic).

Using results obtained above, let us consider quantum brachistochrone problem on the manifolds defined by (18), (19), separately. First, we examine the optimal evolution on the manifold described by equation (18). We take the initial and the final states which belong to the manifold defined by the metric (18) as follows $|\psi_i\rangle = |1\rangle$,

$$|\psi_f\rangle = \begin{pmatrix} \frac{1}{2}(1 + \cos \theta_f) e^{-i\phi_f} \\ \frac{1}{\sqrt{2}} \sin \theta_f \\ \frac{1}{2}(1 - \cos \theta_f) e^{i\phi_f} \end{pmatrix}. \quad (31)$$

The final state is achieved when the angle between the magnetic field and this state is the same as the angle between the magnetic field and the initial

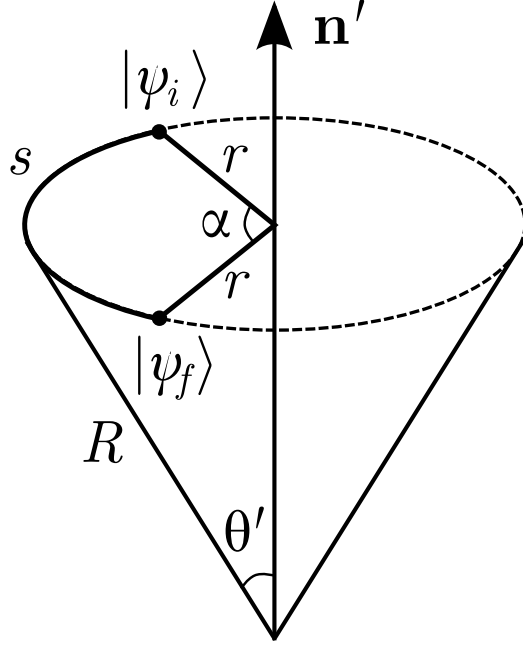


Figure 1: *Identification of the path length of evolution s for a spin system in the magnetic field directed along the unit vector \mathbf{n}' .*

state (see Fig. 1). Then the quantum evolution between two states $|\psi_i\rangle$ and $|\psi_f\rangle$ takes place along the arc of a circle s around the unit vector \mathbf{n}' . From the analysis of the Fig. 1 it is clear that

$$s = \alpha r, \quad (32)$$

where $r = R \sin \theta'$. The initial and the final states are separated by an angle θ_f , therefore the angle α takes the form $\alpha = 2 \arcsin \frac{\sin \frac{\theta_f}{2}}{\sin \theta'}$. Substituting this expression into (32), we obtain the path length of evolution between the initial and the final states on the manifold defined by (18) as follows

$$s = 2R \sin \theta' \arcsin \frac{\sin \frac{\theta_f}{2}}{\sin \theta'}, \quad (33)$$

where $R = \frac{\gamma}{\sqrt{2}}$ is the radius of the manifold being a sphere.

Now it is necessary to calculate the speed of evolution between $|\psi_i\rangle$ and $|\psi_f\rangle$ states. The speed v of quantum evolution is given by the Anandan-Aharonov relation [38] as

$$v = \gamma \sqrt{\langle \psi(t) | (\Delta H)^2 | \psi(t) \rangle}. \quad (34)$$

In order to calculate the speed of evolution on the manifold defined by the equation (18) let us substitute the state (23) and the Hamiltonian (21) into the equation (34)

$$v = \gamma \sqrt{\langle \psi_1(t) | (\Delta H)^2 | \psi_1(t) \rangle} = \gamma \sqrt{\langle 1 | (\Delta H)^2 | 1 \rangle} = \omega R \sin \theta'. \quad (35)$$

Then, using the equation (30) with equations (33) and (35), we find the period of time of evolution between the initial state $|\psi_i\rangle = |1\rangle$ and the final one (31)

$$t = \frac{2}{\omega} \arcsin \frac{\sin \frac{\theta_f}{2}}{\sin \theta'}. \quad (36)$$

The minimal period of time is achieved for $\theta' = \frac{\pi}{2}$. We have

$$t_{min} = \frac{\theta_f}{\omega}. \quad (37)$$

This condition corresponds to the minimal length of path $s_{min} = \gamma \theta_f / \sqrt{2}$ and the maximal speed of evolution $v_{max} = \gamma \omega / \sqrt{2}$. For example, in the case of $\theta_f = \pi$, the minimal path and the minimal time of evolution between two orthogonal states read $s_{min} = \gamma \pi / \sqrt{2}$ and $t_{min} = \pi / \omega$. So, the optimal evolution is achieved for perpendicular orientation of the magnetic field with respect to the initial and the final states. It means that the unit vector which defines direction of the magnetic field takes the form $\mathbf{n}'_{opt} = (-\sin \phi_f, \cos \phi_f, 0)$. The Hamiltonian which provides the optimal evolution takes the following form

$$H_{opt} = \omega \mathbf{S} \cdot \mathbf{n}'_{opt}. \quad (38)$$

The same situation we have in the case of the manifold defined by the equation (19). Here we consider evolution between the initial state $|\psi_i\rangle = |0\rangle$ and the final one

$$|\psi_f\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin \theta_f e^{-i\phi_f} \\ \cos \theta_f \\ \frac{1}{\sqrt{2}} \sin \theta_f e^{i\phi_f} \end{pmatrix}.$$

Then, having performed the same steps as in the previous case, we obtain that the length of the path which the system passes between these states and the speed of evolution of the system are also defined by the equations (33) and (35), respectively. But here the manifold has the following radius $R = \gamma$. As we can see, similarly to the previous case, the period of evolution is defined

by the equation (36). In this case the optimal evolution also corresponds to the perpendicular orientation of the magnetic field to the initial and the final states. Hence the optimal period of time is defined by the equation (37) and the Hamiltonian which provides such evolution is defined by the equation (38). Here the minimal time of evolution between these two orthogonal states is $\pi/2\omega$ because they are separated by an angle $\pi/2$.

5 Generalization for an arbitrary spin

The problem which we considered in the previous sections can be generalized for an arbitrary spin. Namely, what are the geometries of the rotational manifolds which determine the position of the states achieved by the rotation of the eigenstates of the operator of projection of spin- s on the direction \mathbf{n} ? The eigenstate of the operator $\mathbf{S} \cdot \mathbf{n}$ with an eigenvalue m can be represented as follows

$$|\psi_m\rangle = e^{-i\phi S_z} e^{-i\theta S_y} |m\rangle, \quad (39)$$

where \mathbf{S} is the operator of spin- s , \mathbf{n} is defined by the spherical angles θ and ϕ , $|m\rangle$ is the eigenstate of S_z with the eigenvalue m . Here the eigenstates of S_z play the role of the basis vectors. As we can see, the eigenstate (39) is defined by two real parameters θ and ϕ . It is rather difficult to represent the eigenstates of the operator $\mathbf{S} \cdot \mathbf{n}$ for spin-3/2 in the ordinary form and to perform calculations for these states. Therefore, to simplify further calculations we will use the eigenstates of the operator $\mathbf{S} \cdot \mathbf{n}$ written in the form (39).

To obtain metric of the manifold defined by the state (39) let us calculate the following derivatives

$$\begin{aligned} |\psi_{m\ \theta}\rangle &= e^{-i\phi S_z} (-iS_y) e^{-i\theta S_y} |m\rangle, \\ |\psi_{m\ \phi}\rangle &= (-iS_z) e^{-i\phi S_z} e^{-i\theta S_y} |m\rangle. \end{aligned} \quad (40)$$

Then we can write the following scalar products

$$\langle \psi_m | \psi_{m\ \theta} \rangle = -i \langle m | S_y | m \rangle, \quad (41)$$

$$\langle \psi_{m\ \theta} | \psi_{m\ \theta} \rangle = \langle m | S_y^2 | m \rangle, \quad (42)$$

$$\langle \psi_m | \psi_{m\ \phi} \rangle = -i \langle m | e^{i\theta S_y} S_z e^{-i\theta S_y} | m \rangle, \quad (43)$$

$$\langle \psi_{m\ \phi} | \psi_{m\ \phi} \rangle = \langle m | e^{i\theta S_y} S_z^2 e^{-i\theta S_y} | m \rangle, \quad (44)$$

$$\langle \psi_{m\ \theta} | \psi_{m\ \phi} \rangle = \langle m | e^{i\theta S_y} S_y S_z e^{-i\theta S_y} | m \rangle. \quad (45)$$

Having calculated these scalar products we obtain

$$\begin{aligned}
\langle \psi_m | \psi_{m \ \theta} \rangle &= 0, \quad \langle \psi_{m \ \theta} | \psi_{m \ \theta} \rangle = \frac{1}{2} (s + s^2 - m^2) \\
\langle \psi_m | \psi_{m \ \phi} \rangle &= -im \cos \theta, \\
\langle \psi_{m \ \phi} | \psi_{m \ \phi} \rangle &= \frac{1}{2} (s + s^2 - m^2) \sin^2 \theta + m^2 \cos^2 \theta, \\
\langle \psi_{m \ \theta} | \psi_{m \ \phi} \rangle &= i \frac{m}{2} \sin \theta.
\end{aligned} \tag{46}$$

Here we use Baker-Campbell-Hausdorff formula for following operators

$$\begin{aligned}
e^{i\theta S_y} S_z e^{-i\theta S_y} &= S_z \cos \theta - S_x \sin \theta, \\
e^{i\theta S_y} S_z^2 e^{-i\theta S_y} &= e^{i\theta S_y} S_z e^{-i\theta S_y} e^{i\theta S_y} S_z e^{-i\theta S_y} = (e^{i\theta S_y} S_z e^{-i\theta S_y})^2 \\
&= S_x^2 \sin^2 \theta + S_z^2 \cos^2 \theta - (S_x S_z + S_z S_x) \cos \theta \sin \theta.
\end{aligned}$$

Substituting these scalar products into the equation (13), we obtain the components of the metric tensor as follows

$$g_{\theta\theta} = \frac{\gamma^2}{2} (s + s^2 - m^2), \quad g_{\phi\phi} = \frac{\gamma^2}{2} (s + s^2 - m^2) \sin^2 \theta, \quad g_{\theta\phi} = 0. \tag{47}$$

Thus, the Fubini-Study metric of the manifold defined by state (39) is

$$ds_m^2 = \frac{\gamma^2}{2} (s + s^2 - m^2) ((d\theta)^2 + \sin^2 \theta (d\phi)^2). \tag{48}$$

As we see this is the metric of the sphere of radius

$$R = \frac{\gamma}{\sqrt{2}} \sqrt{s + s^2 - m^2}. \tag{49}$$

The Fubini-Study metric of the manifold with $m = -s$ was considered in [34, 35]. We obtain the metric for rotational manifolds with arbitrary m .

From the analysis of (48) it is clear that there exist $s+1$ manifolds for an integer spin and $s+1/2$ manifolds for a half-integer spin. For instance, in the case of spin-3/2 system we have two rotational manifolds with radii $\gamma\sqrt{3}/2$ and $\gamma\sqrt{7}/2$ which correspond $m = \pm 3/2$ and $m = \pm 1/2$, respectively. The same result we obtain directly, using the ordinary form of the eigenstates of the operator $\mathbf{S} \cdot \mathbf{n}$ for spin-3/2.

Let us consider the evolution of spin- s system which takes place on the manifold defined by the metric (48). Hamiltonian which allows us to provide such evolution has the form (21) with the spin- s operator. We take the initial state as follows $|\psi_i\rangle = |m\rangle$ and the final one as follows

$|\psi_f\rangle = e^{-i\phi_f S_z} e^{-i\theta_f S_y} |m\rangle$. Making the same steps as in the case of spin-1 system, we obtain that the length of path between the initial and final states is defined by the equation (33) with radius (49). Using the Anandan-Aharonov relation (34) with the Hamiltonian (21) for spin- s system, we obtain that in the general case the speed of evolution depends on the radius of manifold (49) and on the direction of the magnetic field as follows

$$\begin{aligned} v &= \gamma \sqrt{\langle \psi_m(t) | (\Delta H)^2 | \psi_m(t) \rangle} = \gamma \sqrt{\langle m | (\Delta H)^2 | m \rangle} \\ &= \omega R \sin \theta'. \end{aligned} \quad (50)$$

So, similarly to the case of spin-1 system the optimal evolution happens when the magnetic field is directed perpendicular to the initial and the final states. Then the shortest path between two states which are separated by the angle θ_f is

$$s_{min} = \theta_f R = \theta_f \frac{\gamma}{\sqrt{2}} \sqrt{s + s^2 - m^2} \quad (51)$$

and the maximal speed is

$$v_{max} = \omega R = \omega \frac{\gamma}{\sqrt{2}} \sqrt{s + s^2 - m^2}. \quad (52)$$

Then, using (30) with (51) and (52), we obtain that the minimal time of evolution between two states separated by angle θ_f is determined by equation (37). The optimal Hamiltonian which provides such evolution is defined by (38) with the spin- s operator.

6 Conclusion

Rotations of the eigenstate of the operator S_z with eigenvalue m through an angle θ about the y -axis and an angle ϕ about the z -axis allow us to achieve the eigenstate of the operator of projection of spin- s on the direction $\mathbf{n}(\theta, \phi)$ with the same eigenvalue. This eigenstate belongs to some manifold called rotation manifold defined by two real parameters θ and ϕ . For a spin-1/2 system there exists one rotational manifold which coincides with a two-dimensional quantum space. In the general case, there exist $s + 1$ manifolds for an integer spin and $s + 1/2$ manifolds for a half-integer one. The rotational manifolds for an arbitrary spin- s system (excluding the case with spin-1/2 system) do not coincide with the quantum space of this system. The number of parameters defining each those manifolds is not enough to specify the quantum space of a spin- s system which is given by $4s$ real

parameters. Moreover, rotational manifolds for a spin greater than $1/2$ do not have properties of linear spaces. Linear combination of the states which belong to one of such manifolds does not belong to it.

For the spin-1 system it was shown that there are two rotational manifolds which correspond to the eigenvalues $m = \pm 1$ and $m = 0$ of the operator $\mathbf{S} \cdot \mathbf{n}$. The Fubini-Study metric of the manifold for $m = \pm 1$ is that of the sphere of radius $\gamma/\sqrt{2}$. The orthogonal states correspond to antipodal points on this sphere. The Fubini-Study metric of the another manifold (with $m = 0$) is that of the sphere of radius γ . We showed that this manifold has properties of elliptic geometry. Here orthogonal states are separated by the angle $\pi/2$. These results were generalized for the arbitrary spin s . In the general case the Fubini-Study metric of the rotational manifold which corresponds to the eigenvalue m is that of the sphere with the radius dependent on the value of the spin s and on the value of the spin projection m and is defined by the equation (49). In [34, 35] the Fubini-Study metric of the manifold was considered only for particular case when $m = -s$. We want to emphasize that we obtained metric for rotational manifolds for arbitrary m .

Finally, we considered quantum evolution for the spin-1 system on the rotational manifolds. We solved the quantum brachistochrone problem for the spin-1 system in the magnetic field using geometric properties of the rotational manifolds. We conclude that the optimal evolution happens when the magnetic field is perpendicular to the initial and the final states. The Hamiltonian which provides such evolution is defined by the equation (38). Then the minimal path length between these states which are separated by an angle θ_f is a geodesic line on the rotational manifolds. The minimal period of time of evolution on these manifolds is defined by the equation (37). We obtained similar results in the general case of a spin- s system. Namely, we obtained that the minimal period of time of evolution of a spin- s system in the magnetic field between the initial and the final states separated by an angle θ_f is defined by the equation (37). The Hamiltonian which provides optimal evolution of spin- s system is defined by the equation (38) with spin- s operator. We note that the Hamiltonian of the spin- s system in the magnetic field (excluding the case of spin- $1/2$ system) does not contain enough number of parameters to provide the evolution between two arbitrary quantum states. Therefore, we could not consider the quantum brachistochrone problem for this system on the whole Hilbert space.

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